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NPSOR-91-030

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**A KALMAN FILTER FOR A POISSON SERIES  
WITH COVARIATES AND LAPLACE  
APPROXIMATION INTEGRATION**

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September 1991

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Prepared for:  
Naval Postgraduate School  
Monterey, CA 93943

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This report was prepared in conjunction with research funded under the Naval Postgraduate School Research Council Research Program.

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved  
 OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION <b>UNCLASSIFIED</b>		1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION /AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE		4. PERFORMING ORGANIZATION REPORT NUMBER(S) NPSOR-91-030	
6a. NAME OF PERFORMING ORGANIZATION Naval Postgraduate School		6b. OFFICE SYMBOL (If applicable) OR	
6c. ADDRESS (City, State, and ZIP Code) Monterey, CA 93943		7a. NAME OF MONITORING ORGANIZATION O&MN, Direct Funding	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Naval Postgraduate School		8b. OFFICE SYMBOL (If applicable)	
8c. ADDRESS (City, State, and ZIP Code) Monterey, CA 93943		7b. ADDRESS (City, State, and ZIP Code) Monterey, CA 93943-5006	
9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER		10. SOURCE OF FUNDING NUMBERS	
		PROGRAM ELEMENT NO.	PROJECT NO.
		TASK NO.	WORK UNIT ACCESSION NO.
11. TITLE (Include Security Classification) <i>A Kalman Filter for a Poisson Series with Covariates and Laplace Approximation Integration</i>			
12. PERSONAL AUTHOR(S) Donald P. Gaver and Patricia A. Jacobs			
13a. TYPE OF REPORT Technical Report	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Year, month day) September, 1991	15. PAGE COUNT 47
16. SUPPLEMENTARY NOTATION			
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB-GROUP	
		Poisson time series with covariates; hierarchical model; Kalman filter; Laplace method	
19. ABSTRACT (Continue on reverse if necessary and identify by block number)			
<p>A hierarchical model for a Poisson time series is introduced. The model allows the mean or rate of the Poisson variables to vary slowly in time; it is modeled as the exponential of an AR/1 process. In addition the rate is influenced by a covariate. The Laplace method is used to recursively update some model parameter estimates. Frankly heuristic methods are explored to estimate other of the underlying parameters. The methodology is checked against simulated data with encouraging results.</p>			
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS		21. ABSTRACT SECURITY CLASSIFICATION <b>UNCLASSIFIED</b>	
22a. NAME OF RESPONSIBLE INDIVIDUAL D. P. Gaver		22b. TELEPHONE (Include Area Code) (408) 646-2605	2c. OFFICE SYMBOL OR/Gv



# A KALMAN FILTER FOR A POISSON SERIES WITH COVARIATES AND LAPLACE-APPROXIMATION INTEGRATION

D. P. Gaver  
P. A. Jacobs

## 0. INTRODUCTION

The Poisson model is an initial idealized, but plausible, off-the-shelf tool for representing point-process data of nearly any kind, cf. Feller (1966) and Cox and Lewis (1968). However, to be more descriptive, and even predictive, representation of non-homogeneity in space or time may be needed. For instance the occurrence of rare events in space, such as the occurrence of extreme heights, i.e. above a level of Arctic ice along a transect or encounters with ice keels along a submarine track at constant (deep) depths, may well appear roughly Poisson. A better description of these events may require more detail than a simple mean or rate: some account of regional and seasonal variation could be needed for true accuracy; for instance the intervention of natural gaps along a path in Arctic ice will occur if the latter crosses **leads** (open water amidst an ice pack). For another example, demands for spare parts in a logistics system often are roughly Poisson, or compound Poisson, but with a mean or rate that changes erratically but slowly in time. Demands for communication and computer facilities exhibit a similar temporal pattern, and there are a great many other examples. To summarize, variation in a fundamental Poisson rate or mean is very often encountered in practice; it is possible that if this variation is slow-moving or persistent enough it can be exploited for short-term forecasting.

In this paper we study a model-based procedure for forecasting in the kind of environments described. The model introduced allows the mean or rate of the Poisson process to be itself a random process; the exponential of an AR/1 autoregressive process. In addition the rate is influenced by a covariate. We then recursively update the parameter estimates using an approximation based on the *Laplace method*; cf. de Bruijn (1958). The approach resembles that of Delampady, Yee and Zidek (1991), but frankly heuristic methods are used to estimate certain of the underlying parameters. The methodology is checked against simulated data with encouraging results.

## 1. FORMULATION

Consider the following Poissonian model for count data:

$$P\{Y_t = y_t | \mu_t, x_t, \beta, h_t, y_0, \dots, y_{t-1}\} = \exp\{-h_t e^{x_t \beta + \mu_t}\} \frac{[h_t e^{x_t \beta + \mu_t}]^{y_t}}{y_t!} \quad (1.1)$$

where

$$\mu_t = \alpha \mu_{t-1} + \omega_t \quad (1.2)$$

with  $\{\omega_t\}$  independent normal/Gaussian random variables with mean 0 and variance  $W_t$  independent of  $\{\mu_i; i \leq t-1\}$ ,  $\{Y_i; i \leq t-1\}$ ,  $\{x_i\}$ ,  $\{h_i\}$  and  $\beta$ . Another hierarchical time series model for count data can be found in Harvey et al. (1989).

The purpose of this paper is to suggest a Kalman filter-like procedure to produce successive estimates of  $\beta$  and  $\mu_t$  as new data becomes available. The procedure is based on a Laplace approximation to an integral. A Bayesian approach to a similar problem is being investigated by Delampaday et al.

(1991). A Bayesian approach to time series can be found in West et al. (1989) and for a more recent computational approach in Carlin et al. (1991).

## 2. AN APPROXIMATE UPDATING PROCEDURE

Assume that the posterior distribution of  $(\boldsymbol{\beta}, \boldsymbol{\mu}_{t-1})$  given  $\{y_i, i \leq t-1\}$  is bivariate normal with mean  $(b_{t-1}, m_{t-1})$ ,  $\text{Var}[\boldsymbol{\beta}] = \tau_{t-1}$ ,  $\text{Var}[\boldsymbol{\mu}_{t-1}] = C_{t-1}$  and  $\text{Corr}[\boldsymbol{\beta}, \boldsymbol{\mu}_{t-1}] = \rho_{t-1}$ .

Since it is known that

$$\boldsymbol{\mu}_t = \alpha \boldsymbol{\mu}_{t-1} + \boldsymbol{\omega}_t,$$

the prior distribution of  $(\boldsymbol{\beta}, \boldsymbol{\mu}_t)$  is bivariate normal with mean  $(b_{t-1}, \alpha m_{t-1})$ ,  $\text{Var}[\boldsymbol{\beta}] = \tau_{t-1}$ ,  $\text{Var}[\boldsymbol{\mu}_t] = R_t = \alpha^2 C_{t-1} + W_t$  and  $\text{Corr}[\boldsymbol{\beta}, \boldsymbol{\mu}_t] \equiv r_t = \alpha \rho_{t-1} \sqrt{C_{t-1}/R_t}$ .

The forecast/prediction distribution of  $\mathbf{Y}_t$  in terms of data up to  $t-1$  and the covariate value at  $t$  is

$$\begin{aligned} P \{ \mathbf{Y}_t = y_t | \mathbf{Y}_i = y_i, i \leq t-1, x_i, i \leq t, h_i, i \leq t \} \\ = \int \exp \left\{ -e^{[x_t b + z] h_t} \right\} \frac{\left[ e^{[x_t b + z] h_t} \right]^{y_t}}{y_t!} \frac{1}{2\pi \sqrt{\tau_{t-1} R_t} (1 - r_t^2)} \\ \times \exp \left\{ -\frac{1}{2} (1 - r_t^2)^{-1} \left[ \frac{(b - b_t^p)^2}{\tau_{t-1}} - \frac{2r_t (b - b_t^p)(z - m_t^p)}{\sqrt{\tau_{t-1}} \sqrt{R_t}} + \frac{(z - m_t^p)^2}{R_t} \right] \right\} dz db. \quad (21) \end{aligned}$$

where  $b_t^p = b_{t-1}$  and  $m_t^p = \alpha m_{t-1}$ .

We now approximate the integral by the Laplace method; cf. Easton (1991), Cox and Hinkley (1974), de Bruijn (1958). Let the exponent of the integrand be

$$g(b, z) = -e^{x_t b + z} h_t + y_t [x_t b + z] + y_t \ln h_t \cdot \frac{1}{2} (1 - r_t^2)^{-1} \\ \times \left[ \frac{(b - b_t^p)^2}{\tau_{t-1}} - \frac{2r_t}{\sqrt{\tau_{t-1} R_t}} (b - b_t^p)(z - m_t^p) + \frac{(z - m_t^p)^2}{R_t} \right] + K \quad (2.2)$$

where  $K$  is a constant.

$$\frac{\partial}{\partial z} g(b, z) = y_t \cdot e^{x_t b + z} h_t - \frac{1}{1 - r_t^2} \left[ \frac{(z - m_t^p)}{R_t} - r_t \frac{(b - b_t^p)}{\sqrt{\tau_{t-1} \sqrt{R_t}}} \right] \quad (2.3)$$

$$\frac{\partial^2}{\partial z^2} g(b, z) = -e^{(x_t b + z)} h_t - \frac{1}{1 - r_t^2} \frac{1}{R_t} \quad (2.4)$$

$$\frac{\partial}{\partial b} g(b, z) = (y_t \cdot e^{x_t b + z} h_t) x_t - [1 - r_t^2]^{-1} \left[ \frac{(b - b_t^p)}{\tau_{t-1}} - \frac{r_t (z - m_t^p)}{\sqrt{\tau_{t-1} \sqrt{R_t}}} \right] \quad (2.5)$$

$$\frac{\partial^2}{\partial b^2} g(b, z) = -e^{(x_t b + z)} h_t x_t^2 - [1 - r_t^2]^{-1} \frac{1}{\tau_{t-1}} \quad (2.6)$$

$$\frac{\partial^2}{\partial b \partial z} g(b, z) = -e^{(x_t b + z)} h_t x_t + (r_t) (1 - r_t^2)^{-1} (\tau_{t-1} R_t)^{-0.5} \quad (2.7)$$

Use a Newton procedure to solve the system of equations

$$0 = \frac{\partial}{\partial z} g \\ 0 = \frac{\partial}{\partial b} g \quad (2.8)$$

for  $m_t$  and  $b_t$ . Solve for  $\tau_t$ ,  $C_t$  and  $\rho_t$  using the relation



$$\begin{bmatrix} \tau_t & \rho_t \sqrt{C_t \tau_t} \\ \rho_t \sqrt{C_t \tau_t} & C_t \end{bmatrix}^{-1} = - \begin{bmatrix} \frac{\partial^2}{\partial b^2} g \Big|_{m_t, b_t} & \frac{\partial^2}{\partial b \partial z} g \Big|_{m_t, b_t} \\ \frac{\partial^2}{\partial b \partial z} g \Big|_{m_t, b_t} & \frac{\partial^2}{\partial z^2} g \Big|_{m_t, b_t} \end{bmatrix}. \quad (2.9)$$

The posterior distribution for  $(\boldsymbol{\beta}, \boldsymbol{\mu}_t)$  given  $y_t, D_{t-1}$  is approximated by a bivariate normal distribution with mean  $(b_t, m_t)$  and variance-covariance matrix

$$\Sigma_t = \begin{bmatrix} \tau_t & \rho_t \sqrt{C_t \tau_t} \\ \rho_t \sqrt{C_t \tau_t} & C_t \end{bmatrix}. \quad (2.10)$$

## 2.1 Summary of the Newton Procedure

0. Start with estimates of the parameters of the prior bivariate distribution of  $(\boldsymbol{\beta}, \boldsymbol{\mu}_t)$ ; that is, estimates of the mean  $(b_{t-1}, \alpha m_{t-1})$ ,  $\text{Corr}(\boldsymbol{\beta}, \boldsymbol{\mu}_t) = r_t$ ,  $\text{Var}(\boldsymbol{\beta}) = \tau_{t-1}$ ,  $\text{Var}[\boldsymbol{\mu}_t] = R_t$ . Set  $b_t^0 = b_{t-1}$ ,  $m_t^0 = \alpha m_{t-1}$ .

1. Solve the system of linear equations (in  $b$  and  $m$ )

$$\begin{aligned} 0 &= \frac{\partial}{\partial z} g(b_t^0, m_t^0) + \frac{\partial^2}{\partial b \partial z} g(b_t^0, m_t^0)[b - b_t^0] + \frac{\partial^2}{\partial z^2} g(b_t^0, m_t^0)[m - m_t^0] \\ 0 &= \frac{\partial}{\partial b} g(b_t^0, m_t^0) + \frac{\partial^2}{\partial b^2} g(b_t^0, m_t^0)[b - b_t^0] + \frac{\partial^2}{\partial z \partial b} g(b_t^0, m_t^0)[m - m_t^0] \end{aligned}$$

for  $b, m$ .

2. If  $\max\left[\left(\frac{|b_t^0 - b|}{b}\right), \left(\frac{|m_t^0 - m|}{m}\right)\right] < 0.001$ , set  $m_t = m$  and  $b_t = b$  and go to 3.

3. Otherwise set  $b_t^0 = b$  and  $m_t^0 = m$  and return to Step 1 unless Step 1 has been returned to 49 times in which case set  $m_t = m_t^0$  and  $b_t = b_t^0$ ; go to 3.

3. Return  $(b_t, m_t)$  as the estimate of the posterior mean of the bivariate normal distribution of  $(\boldsymbol{\beta}, \boldsymbol{\mu}_t)$ .

4. To find the variance-covariance matrix of the posterior distribution of  $(\beta, \mu_t)$  evaluate

$$\begin{bmatrix} \frac{\partial^2}{\partial b^2} g(b_t, m_t) & \frac{\partial^2}{\partial z \partial b} g(b_t, m_t) \\ \frac{\partial^2}{\partial z \partial b} g(b_t, m_t) & \frac{\partial^2}{\partial z^2} g(b_t, m_t) \end{bmatrix}^{-1};$$

set it equal to

$$\begin{bmatrix} \tau_t & \rho_t \sqrt{\tau_t} \sqrt{C_t} \\ \rho_t \sqrt{\tau_t} \sqrt{C_t} & C_t \end{bmatrix}$$

and solve for  $\tau_t$ ,  $\rho_t$  and  $C_t$

## 2.2 Summary of Kalman Procedure

In summary the approximate Kalman procedure is as follows

0. Start with the parameters of the (approximate) posterior bivariate normal distribution of  $(\beta, \mu_{t-1})$  having mean  $(b_{t-1}, m_{t-1})$ ,  $\text{Corr}(\beta, \mu_{t-1}) = \rho_{t-1}$ ,  $\text{Var}[\beta] = \tau_{t-1}$   $\text{Var}[\mu_{t-1}] = C_{t-1}$ .

1. Update  $\text{Corr}(\beta, \mu_{t-1})$  and  $\text{Var}[\mu_{t-1}]$  using (1.2) to obtain the prior bivariate normal distribution of  $(\beta, \mu_t)$  having mean  $(b_{t-1}, \alpha m_{t-1})$ ,  $\text{Var}[\beta] = \tau_{t-1}$ ,  $\text{Var}[\mu_t] = R_t = \alpha^2 C_{t-1} + W_t$ ,  $\text{Corr}(\beta, \mu_t) = r_t = \alpha \rho_{t-1} \sqrt{C_{t-1}/R_t}$ .

2. Observe the Poisson count  $y_t$

3. Invoke the Newton procedure of Section 2.1 to obtain estimates of the parameters of the approximate posterior distribution of  $(\beta, \mu_t)$  given past observations and the new observation  $y_t$ .

To obtain moments for the Poisson mean  $\lambda_t = h_t \exp\{x_t \beta + \mu_t\}$  note that since the distribution of  $(\beta_t, \mu_t)$  is being approximated by a bivariate normal distribution, the posterior moments of  $\lambda_t$  are approximately

$$E[\lambda_t] \approx \exp\left\{x_t b_t + m_t + \frac{1}{2}\left[x_t^2 \tau_t + C_t + 2x_t \rho_t \sqrt{\tau_t} \sqrt{C_t}\right]\right\} h_t$$

$$\text{Var}[\lambda_t] \approx h_t^2 \exp\left\{2(x_t b_t + m_t) + \left[x_t^2 \tau_t + C_t + 2x_t \rho_t \sqrt{\tau_t} \sqrt{C_t}\right]\right\} \\ \times \left[\exp\left\{x_t^2 \tau_t + C_t + 2x_t \rho_t \sqrt{\tau_t} \sqrt{C_t}\right\} - 1\right].$$

The estimate of  $\lambda_t$  is  $\hat{\lambda}_t = e^{b_t x_t + m_t} h_t$ .

### 2.3 A Simulation Example

In the example  $\beta = 0.5$ ,  $\{x_t\}$  takes the values  $\{0.25, .5, 1\}$  over and over;  $h_t \equiv 1$ .  $\{\omega_t\}$  are iid normal with mean 0 and variance 0.25 and  $\alpha = 0.5$ . Given  $\beta$  and  $\mu_t$ ,  $Y_t$  has a Poisson distribution with mean  $\lambda_t = e^{x_t \beta + \mu_t}$ . The simulation starts with  $\mu_0$  drawn from the stationary distribution of  $\{\mu_t\}$  a normal distribution with mean 0 and variance  $W/(1-\alpha^2) = \frac{1}{3}$ . Successive  $\mu_t$  are computed as

$$\mu_t = \alpha \mu_{t-1} + \omega_t;$$

the  $\lambda_t$  is computed and a Poisson random number with mean  $\lambda_t$  is then generated. The simulation data are the Poisson counts and  $\{x_t\}$ ;  $t = 1, \dots, 100$ . The random numbers were generated using LLRANDOM II; cf. Lewis et al. (1981).

At time 0, the Newton procedure is initialized at  $\mu_0 = 0, \beta = 0, \rho_0 = r_0 = 0, \alpha = 0.5, W = 0.25, C_0 = 0, r_1 = 0.25, \tau_0 = 1$ ; note the  $\alpha$  and  $W$  are assumed known. The data point  $y_1$  is observed and the Newton procedure is used to find the posterior moments  $[m_1, b_1, \rho_1, C_1, \tau_1]$ .

The data point  $y_2$  is the observed and the Newton procedure is started with  $[m_1, b_1, r_1, C_1 + \frac{1}{4}, \tau_1]$  and used to find  $[m_2, b_2, \rho_2, C_2, \tau_2]$ , etc.

Results of the simulation appear in Figures 1-3. In Figure 1 the count data  $y_t$  appears along with the true  $\lambda_t$  (dotted line) and the estimated  $\lambda_t$  (solid line). In Figure 2 the true value of  $\mu_t$  (dotted line) and estimated value of  $\mu_t$  (solid line) appear. The estimated values of the standard deviation of  $\hat{\mu}_t, \sqrt{C_t}$ , also appear in Figure 2. The estimated value of  $\beta, \beta$ 's estimated standard deviation  $\sqrt{\tau_t}$ , and the estimated value of  $\rho_t$  appear in Figure 3. Figures 1-3 indicate that the procedure performs satisfactorily. The apparent oscillation in some of the figures, (particularly  $\hat{\rho}_t$ ), is due to the cyclic nature of  $\{x_t\}$ . Not surprisingly, the estimated values of  $\lambda_t$  and  $\mu_t$  are less variable than the true values. They appear to be practically acceptable.

### 3. APPROXIMATE KALMAN PROCEDURES WHICH INCORPORATE ESTIMATES OF THE AUTOREGRESSIVE PARAMETERS $\alpha$ AND $W$ : NAIVE MOMENT ESTIMATORS

In this section we will assume that  $\{\omega_t\}$  are independent identically distributed normal/Gaussian random variables with mean 0 and variance  $W$  with  $\omega_t$  independent of  $\{\mu_i; i \leq t-1\}, \{Y_i; i \leq t-1\}, \{h_t\}$  and  $\beta$ .

#### 3.1 Estimates of the Autoregressive Parameters $\alpha$ and $W$

If  $\{\mu_i; t \leq T\}$  were observable, then one could estimate  $\alpha$  and  $W$  by maximum likelihood; that is, the likelihood function is

$$L(W, \alpha) = \prod_{t=1}^T \frac{1}{\sqrt{2\pi W}} \exp\left\{-\frac{1}{2}(\mu_t - \alpha\mu_{t-1})^2 \frac{1}{W}\right\} \quad (3.1)$$

or

$$\ln L(W, \alpha) = l(W, \alpha) = \frac{1}{2} \sum_{t=1}^T (\mu_t - \alpha \mu_{t-1})^2 \frac{1}{W} - \frac{T}{2} \ln W - \frac{T}{2} \ln 2\pi. \quad (3.2)$$

Now

$$\frac{\partial l}{\partial \alpha} = - \sum_{t=1}^T (\mu_t - \alpha \mu_{t-1}) \mu_{t-1} \frac{1}{W} = 0 \quad (3.3)$$

gives

$$\hat{\alpha}(T) = \sum_{t=1}^T \mu_t \mu_{t-1} / \sum_{t=1}^T \mu_{t-1}^2 \quad (3.4)$$

and

$$\frac{\partial l}{\partial W} = \frac{1}{2} \frac{1}{W^2} \sum_{t=1}^T (\mu_t - \alpha \mu_{t-1})^2 - \frac{T}{2} \frac{1}{W} = 0 \quad (3.5)$$

from which

$$\hat{W}(T) = \frac{1}{T} \sum_{t=1}^T (\mu_t - \hat{\alpha}(T) \mu_{t-1})^2. \quad (3.6)$$

Unfortunately  $\mu_t$  is not observable. One possible estimate of  $\mu_t$  is the posterior mean  $m_t$  of subsections (2.1) and (2.2). In this case the corresponding estimators of  $\alpha$  and  $W$  are

$$\hat{\alpha}_m(T) = \sum_{t=1}^T m_t m_{t-1} / \sum_{t=1}^T m_{t-1}^2 \quad (3.7)$$

$$\hat{W}_m(T) = \frac{1}{T} \sum_{t=1}^T (m_t - \hat{\alpha}_m(T) m_{t-1})^2. \quad (3.8)$$

Another estimate of  $\mu_t$  is

$$\hat{\mu}_t(y) = \ln \left[ y_t + \frac{1}{2} \right] \cdot x_t b_t \quad (3.9)$$

where  $b_t$  is the mean of the approximate posterior distribution of  $\beta$  given  $\{y_s; s \leq t\}$ . The corresponding estimates of  $\alpha$  and  $W$  are

$$\hat{\alpha}_y(T) = \frac{\sum_{t=1}^T (\hat{\mu}_t(y) \hat{\mu}_{t-1}(y))}{\sum_{t=1}^T \hat{\mu}_t(y)^2} \quad (3.10)$$

and

$$\hat{W}_y(T) = \frac{1}{T} \sum_{t=1}^T (\hat{\mu}_t(y) - \hat{\alpha}_y(T) \hat{\mu}_{t-1}(y))^2. \quad (3.11)$$

The estimates (3.7) – (3.11) can be recomputed at every time  $T$ . Other similar estimates can be obtained by not recomputing the estimates at every time  $T$ ; for example, choose an integer  $\delta > 1$ , put

$$\hat{\alpha}(t; \delta) = \hat{\alpha}((n-1)\delta; \delta) \quad (3.12)$$

and

$$\hat{W}(t; \delta) = \hat{W}((n-1)\delta; \delta) \quad (3.13)$$

if  $(n-1)\delta < t < n\delta$ ;

$$\hat{\alpha}(n\delta; \delta) = \frac{\sum_{t=1}^{n\delta} \hat{\mu}_t \hat{\mu}_{t-1}}{\sum_{t=1}^{n\delta} \hat{\mu}_{t-1}^2} \quad (3.14)$$

$$\hat{W}(n\delta; \delta) = \frac{1}{n\delta} \sum_{t=1}^{n\delta} (\hat{\mu}_t - \hat{\alpha}(n\delta; \delta) \hat{\mu}_{t-1})^2 \quad (3.15)$$

where  $\mu_t$  is an estimate of  $\mu_t$ . Another possibility is to use a window of times to compute estimates of  $\alpha$  and  $W$ .

The following is a summary of the Kalman procedure of Section 2 with the addition of estimation of  $\alpha$  and  $W$ .

### 3.2 Summary of the Kalman Procedure with Estimation of $\alpha$ and $W$

0. Start with the parameters of the (approximate) posterior bivariate normal distribution of  $(\boldsymbol{\beta}, \boldsymbol{\mu}_{t-1})$  having mean  $(b_{t-1}, m_{t-1})$ ,  $\text{Corr}(\boldsymbol{\beta}, \boldsymbol{\mu}_{t-1}) = \rho_{t-1}$ ,  $\text{Var}[\boldsymbol{\beta}] = \tau_{t-1}$ ,  $\text{Var}[\boldsymbol{\mu}_{t-1}] = C_{t-1}$ .
1. Update  $\text{Corr}(\boldsymbol{\beta}, \boldsymbol{\mu}_{t-1})$  and  $\text{Var}[\boldsymbol{\mu}_{t-1}]$  using (1.2) and the current estimates of  $\alpha$  and  $W$  to obtain the prior bivariate normal distribution of  $(\boldsymbol{\beta}, \boldsymbol{\mu}_t)$  having mean  $(b_{t-1}, \hat{\alpha}_{t-1} m_{t-1})$ ,  $\text{Var}[\boldsymbol{\beta}] = \tau_{t-1}$ ,  $\text{Var}[\boldsymbol{\mu}_t] \equiv R_t = \hat{\alpha}_{t-1}^2 C_{t-1} + \hat{W}_{t-1}$ ,  $\text{Corr}(\boldsymbol{\beta}, \boldsymbol{\mu}_t) = \hat{\alpha}_{t-1} \rho_{t-1} \sqrt{C_{t-1}/R_t}$ .
2. Observe the Poisson count  $y_t$ .
3. Invoke the Newton procedure of Subsection 2.1 to obtain estimates of the parameters of the posterior bivariate normal distribution of  $(\boldsymbol{\beta}, \boldsymbol{\mu}_t)$  given past observations and the new observation  $y_t$ .
4. Compute new estimates of  $\alpha$  and  $W$ :  $\hat{\alpha}_t, \hat{W}_t$ .
5. Return to 1.

### 3.3 Results of Simulation Experiments

Various simulation experiments were carried out using the procedures outlined in this section to estimate  $\alpha$  and  $W$ . One striking phenomenon was that if the count data has a long run of zeroes, then the Kalman procedure reacts by estimating  $\alpha > 1$  and trying to estimate  $\boldsymbol{\mu}_t$  to be a negative number large in absolute value. This tendency caused the numerical Newton procedure discussed in Section 2 to become unstable, and rendered the predicted value of  $\boldsymbol{\lambda}_t$  very slow to respond to positive counts when they did eventually occur. This behavior was ameliorated by arbitrarily putting a lower bound of  $-2$  on  $\hat{\boldsymbol{\mu}}_t$ . A run of zero counts could also make the estimate of  $W$  become close to zero. A small value of  $W$  makes the Kalman filter unresponsive to count changes. This behavior was mitigated by recomputing

estimates of  $\alpha$  and  $W$  every  $\delta = 10$  time units rather than at every time, and using all previous times  $t$  as in (3.7)–(3.8). The estimates of  $\alpha$  and  $W$  were also not computed if the last 10 observed counts were zero. Further a lower bound of 0.1 was arbitrarily set on estimates of  $W$ . Such heuristics are not claimed to be optimal, but are needed to allow the procedures to behave suitably. A search for more systematic procedures can be carried out in future work.

The following are empirical observations. The estimates of  $\alpha$  and  $W$  using  $\hat{\mu}(t) = m_t$  tends to produce practically acceptable estimates of  $\alpha$  but underestimates  $W$ —it tends to make it very small. This behavior is not unexpected since  $m_t$  can be thought of as a smoothed estimate of the  $\mu_t$  and so will have a smaller variance than  $\mu_t$ . This underestimation of  $W$  is not improved by using  $\hat{\mu}_t = \ln\left[y_t + \frac{1}{2}\right]$ . Neither does using “windows” of length  $\bar{\delta}$  seem to solve the problem. Other procedures for estimating  $\alpha$  and  $W$  are explored in the following two sections.

#### 4. ESTIMATION OF $\alpha$ AND $W$ FOR THE POISSON/NORMAL MODEL: THE FREEMAN-TUKEY TRANSFORMATION

This section reports another possible approach to the estimation of the autoregressive parameters for a Poisson/normal model.

The model is as before

$$\mu_{t+1} = \alpha\mu_t + \omega_{t+1} \quad (4.1)$$

where  $\{\omega_t\}$  are iid normal with mean 0 and variance  $W$ ;

$$P\{Y_t = y | \mu_t, x_t\} = \exp\{-e^{\mu_t + \beta x_t h_t}\} \frac{[e^{\mu_t + \beta x_t h_t}]^y}{y!}. \quad (4.2)$$



For simplicity of notation we will assume  $h_t \equiv 1$ .

If  $W$  and  $\alpha$  are known, then an approximate Kalman procedure has been given previously. The issue here is the estimation of  $\alpha$  and  $W$ .

Let

$$g(x) = \sqrt{x} + \sqrt{x+1} \quad (4.3)$$

and

$$Z_t = g(Y_t). \quad (4.4)$$

The conditional distribution of  $Z_t$  given  $\mu_t$  is approximately normal with mean  $k(\mu_t + x_t\beta)$ , where  $k(x) = [4e^x + 1]^{0.5}$ , and variance 1; c.f. Freeman and Tukey (1949, 1950) and Bishop, Fienberg and Holland (1975). Thus if  $\{V_{t+1}\}$  are iid standard normal, given  $D_t = (Y_0, Y_1, \dots, Y_t)$

$$\begin{aligned} Z_{t+1} &\approx k(\mu_{t+1} + x_{t+1}\beta) + V_{t+1} \\ &= k(\alpha\mu_t + \omega_{t+1} + x_{t+1}\beta) + V_{t+1} \\ &= k(\alpha m_t + \alpha(\mu_t - m_t) + \omega_{t+1} + x_{t+1}b_t + x_{t+1}(\beta - b_t)) + V_{t+1} \\ &\approx k(\alpha m_t + x_{t+1}b_t) + k'(\alpha m_t + x_{t+1}b_t)[\alpha(\mu_t - m_t) + x_{t+1}(\beta - b_t) + \omega_{t+1}] + V_{t+1}. \end{aligned} \quad (4.5)$$

Hence,

$$E[Z_{t+1}|D_t] \approx k(\alpha m_t + x_{t+1}b_t) \quad (4.6)$$

$$\text{Var}[Z_{t+1}|D_t] \approx [k'(\alpha m_t + x_{t+1}b_t)]^2 [\alpha^2 C_t + x_{t+1}^2 \tau_t + 2\alpha x_{t+1} \rho_t \sqrt{C_t} \sqrt{\tau_t} + W] + 1. \quad (4.7)$$

An approximate joint distribution of the transformed observations is

$$\begin{aligned}
& P\{Z_1 \in dz_1, Z_2 \in dz_2, \dots, Z_{T+1} \in dz_{T+1}\} \\
&= P\{Z_1 \in dz_1\}P\{Z_2 \in dz_2|Z_1 = z_1\} \times \dots \times P\{Z_{T+1} \in dz_{T+1}|Z_1 = z_1, \dots, Z_T = z_T\} \\
&= P\{Y_1 = dy_1\}P\{Z_2 \in dz_2|Y_1 = y_1\} \times \dots \times P\{Z_{T+1} \in dz_{T+1}|Y_1 = y_1, \dots, Y_T = y_T\} \\
&\approx P\{Y_1 = y_1\} \prod_{t=1}^T \frac{1}{\sqrt{2\pi}f_t(\alpha, W)^{0.5}} \exp\left\{-\frac{1}{2}[z_{t+1} - k(\alpha m_t + x_{t+1}b_t)]^2 / f_t(\alpha, W)\right\} \quad (4.8)
\end{aligned}$$

where

$$f_t(\alpha, W) = [\alpha^2 C_t + x_{t+1}^2 \tau_t + 2\alpha \rho_t x_{t+1} \sqrt{C_t} \sqrt{\tau_t} + W][k'(\alpha m_t + x_{t+1}b_t)]^2 + 1$$

Hence an approximate ln-likelihood function is (up to addition of constants),

$$l(\alpha, W) = \sum_{t=1}^T -\frac{1}{2} \ln f_t(\alpha, W) - \frac{1}{2} (z_{t+1} - k(\alpha m_t + x_{t+1}b_t))^2 f_t(\alpha, W)^{-1}. \quad (4.9)$$

Note that

$$\begin{aligned}
k(x) &= [4e^x + 1]^{0.5}; \\
k'(x) &= \frac{1}{2} [4e^x + 1]^{-0.5} 4e^x = \frac{2e^x}{\sqrt{4e^x + 1}}; \\
k''(x) &= \frac{2e^x [2e^x + 1]}{[4e^x + 1]^{1.5}}. \quad (4.10)
\end{aligned}$$

Further,

$$\frac{\partial}{\partial W} f_t(\alpha, W) = [k'(\alpha m_t + x_{t+1} b_t)]^2 \quad (4.11)$$

$$\begin{aligned} \frac{\partial}{\partial \alpha} f_t(\alpha, W) &= [2\alpha C_t + 2x_{t+1} \rho_t \sqrt{C_t} \sqrt{\tau_t}] [k'(\alpha m_t + x_{t+1} b_t)]^2 \\ &+ [\alpha^2 C_t + x_{t+1}^2 \tau_t + 2\alpha \rho_t x_{t+1} \sqrt{C_t} \sqrt{\tau_t} + W] 2[k'(\alpha m_t + x_{t+1} b_t)] \\ &\times k''(\alpha m_t + x_{t+1} b_t) m_t. \end{aligned} \quad (4.12)$$

Differentiating (4.9) with respect to  $\alpha$  and  $W$  we obtain

$$\frac{\partial l}{\partial \alpha} = \sum_{t=1}^T -\frac{1}{2} \frac{\frac{\partial}{\partial \alpha} f_t(\alpha, w)}{f_t(\alpha, w)} \left[ 1 - \frac{[z_{t+1} - k(\alpha m_t + x_{t+1} b_t)]^2}{f_t(\alpha, W)} \right] \quad (4.13)$$

$$+ (z_{t+1} - k(\alpha m_t + x_{t+1} b_t)) k'(\alpha m_t + x_{t+1} b_t) m_t f_t(\alpha, W)^{-1}$$

$$\frac{\partial l}{\partial W} = \sum_{t=1}^T -\frac{1}{2} \frac{1}{f_t(\alpha, W)} \frac{\partial}{\partial W} f_t(\alpha, W) + \frac{1}{2} [z_{t+1} - k(\alpha m_t + x_{t+1} b_t)]^2 \frac{\frac{\partial}{\partial W} f_t(\alpha, W)}{f_t(\alpha, W)^2} \quad (4.14)$$

$$= \sum_{t=1}^T -\frac{1}{2} \frac{1}{f_t(\alpha, W)} \frac{\partial}{\partial W} f_t(\alpha, W) \left[ 1 - \frac{[z_{t+1} - k(\alpha m_t + x_{t+1} b_t)]^2}{f_t(\alpha, W)} \right]$$

$$E \left[ \frac{\partial^2 l}{\partial \alpha^2} \right] = -\sum_{t=1}^T \left\{ \frac{1}{2} \left[ \frac{\frac{\partial}{\partial \alpha} f_t(\alpha, W)}{f_t(\alpha, W)} \right]^2 + \frac{[k'(\alpha m_t + x_{t+1} b_t) m_t]^2}{f_t(\alpha, W)} \right\}. \quad (4.15)$$

$$E \left[ \frac{\partial^2 l}{\partial^2 W} \right] = -\sum_{t=1}^T \frac{1}{2} \left[ \frac{\frac{\partial}{\partial W} f_t(\alpha, W)}{f_t(\alpha, W)} \right]^2. \quad (4.16)$$

$$E \left[ \frac{\partial^2 l}{\partial \alpha \partial W} \right] = - \sum_{t=1}^T \frac{1}{2} \left[ \frac{\frac{\partial}{\partial W} f_t(\alpha, W) \frac{\partial}{\partial \alpha} f_t(\alpha, W)}{f_t(\alpha, W)^2} \right] \quad (4.17)$$

To avoid difficulties with a Newton procedure involving the restricted range of  $W > 0$ , we will reparameterize;  $W = e^\gamma$ . In this case

$$\frac{\partial l}{\partial \gamma} = \frac{\partial l}{\partial W} e^\gamma$$

$$E \left[ \frac{\partial^2 l}{\partial \gamma^2} \right] = E \left[ \frac{\partial^2 l}{\partial W^2} \right] e^{2\gamma}$$

$$E \left[ \frac{\partial^2 l}{\partial \alpha \partial \gamma} \right] = E \left[ \frac{\partial^2 l}{\partial W \partial \alpha} \right] e^\gamma$$

and  $W$  is replaced by  $e^\gamma$  in all the expressions (4.12) – (4.17).

A Kalman procedure with this method of estimating  $\alpha$  and  $W$  is similar to that of Section 3.2 except that Step 1 is replaced by the numerical solution of the system of equations,

$$\frac{\partial l}{\partial \alpha} = 0 \quad (4.18)$$

$$\frac{\partial l}{\partial \gamma} = 0 \quad (4.19)$$

by a Newton procedure using Fisher's scoring method. Set  $\hat{W} = e^{\hat{\gamma}}$  where  $\hat{\gamma}$  is the solution. Occasionally, there are numerical problems with the Newton procedure. In these cases search is used to find estimates of  $\alpha$  and  $W$ .

A summary of the Newton (with default search) procedure is as follows. The procedure starts with the previously estimated  $\alpha$  and  $\gamma$  called  $\alpha_0$  and  $\gamma_0$ .

0. Set  $B = 0$ .

1. Compute  $f(\partial l, \partial \alpha)$ ,  $f(\partial l, \partial \gamma)$ ,  $E \cdot b \cdot c \cdot [(f(\partial^2 l, \partial \alpha^2))$ ,  $E \left[ \frac{\partial^2 l}{\partial \gamma^2} \right]$ , and  $E \left[ \frac{\partial^2 l}{\partial \alpha \partial \gamma} \right]$  using  $\alpha_0$  and  $\gamma_0$ . If  $\left| E \left[ \frac{\partial^2 l}{\partial \alpha \partial \gamma} \right] \right| < 0.001$  or  $E \left[ \frac{\partial^2 l}{\partial \gamma^2} \right] < 0.0001$  go to 2', otherwise go to 2.

2. Solve the system of linear equations (in terms  $\alpha$  and  $\gamma$ )

$$0 = \frac{\partial l}{\partial \alpha} + E \left[ \frac{\partial^2 l}{\partial \alpha^2} \right] (\alpha - \alpha_0) + E \left[ \frac{\partial^2 l}{\partial \alpha \partial \gamma} \right] (\gamma - \gamma_0)$$

$$0 = \frac{\partial l}{\partial \gamma} + E \left[ \frac{\partial^2 l}{\partial \alpha \partial \gamma} \right] (\alpha - \alpha_0) + E \left[ \frac{\partial^2 l}{\partial \gamma^2} \right] (\gamma - \gamma_0)$$

for  $\alpha_n$  and  $\gamma_n$ . Go to 3.

- 2'. Set  $\gamma_n = \gamma_0$ . If  $\frac{\partial l}{\partial \alpha}$  is of the same sign for both the current and previous value of  $\alpha$ , set

$$\alpha_n = \alpha_0 - \frac{\partial l}{\partial \alpha} / E \left[ \frac{\partial^2 l}{\partial \alpha^2} \right]$$

and go to 3. If  $\frac{\partial l}{\partial \alpha}$  is of different signs for the current and previous values for  $\alpha$ , then a golden section search is used to solve the equation

$$0 = \frac{\partial l}{\partial \alpha};$$

set  $\alpha_n$  equal to the first value of  $\alpha$  for which  $\left| \frac{\partial l}{\partial \alpha} \right| < 0.1$ ; if the number of iterations for this one-dimensional search is greater than 50, go to 4.

3. If  $\max \left\{ \left| \frac{\partial l}{\partial \alpha} \right| (\alpha_n), \left| \frac{\partial l}{\partial \alpha} \right| (\gamma_n) \right\} < 0.1$  stop and take  $\alpha_n$  and  $\gamma_n$  as the estimated values.
- 3'. If  $|\alpha_n| < 5$  and  $\gamma_n < 5$ , go to 5.

- 3''. If  $|\alpha_n| > 5$  or  $\gamma_n > 5$ , let  $B = B + 1$ . If  $B = 1$ , go to 4. If  $B = 2$ , go to 4'. If  $B \geq 3$ , to 4''.
4. Do a two-dimensional search of the ln-likelihood function (4.9) parameterized in terms of  $\alpha$  and  $W$  for a maximizing value. The grid for  $\alpha$  is  $[-4,4]$  in steps of 0.2; the grid for  $W$  is  $[0,8]$  in steps of 0.2. If

$$\max \left[ \left| \frac{\partial l}{\partial \alpha}(\alpha_n) \right|, \left| \frac{\partial l}{\partial \gamma}(\ln W) \right| \right] < 0.1$$

return  $\alpha_n$  and  $\gamma_n = \ln W$  as the estimated values; otherwise set  $\alpha_0 = \alpha_n$  and  $\gamma_0 = \ln W$  and return to 1.

- 4'. Same as 4 except the grid for  $\alpha$  is  $[-8,8]$  in steps of 0.1 and the grid for  $W$  is  $[0.1,8]$  in steps of 0.1 with the additional points 0.000001, 0.00001, 0.0001, 0.001, 0.01.
- 4''.  $\gamma$  is set equal to its previous value and a golden section search for the zero of  $\frac{\partial l}{\partial \alpha}$  is done as in 2'.
5. Set  $\alpha_0 = \alpha_n$  and  $\gamma_0 = \gamma_n$  and return to 1 if the number of iterations is less than 50. If the number of iterations is greater than 50 set  $B = B + 1$ . If  $B = 1$ , go to 4. If  $B \geq 6$ , return  $\alpha_n$  and  $\gamma_n$  as the estimated values. If  $1 < B \leq 5$ , go to 6.
6. Compute  $\left| \frac{\partial l}{\partial \alpha}(\alpha_n) \right|$  and  $\left| \frac{\partial l}{\partial \gamma}(\gamma_n) \right|$ . If  $\left| \frac{\partial l}{\partial \alpha}(\alpha_n) \right| < \left| \frac{\partial l}{\partial \gamma}(\gamma_n) \right|$  go to 6'; otherwise go to 6''.
- 6'. Fix  $\alpha_n = \alpha_0$  and do a search over the interval  $[-4B, 4B]$  for that  $\gamma$  for which

$$0 = \frac{\partial l}{\partial \gamma}.$$

Set  $\gamma_n$  equal to the first value of  $\gamma$  for which  $\left| \frac{\partial l}{\partial \gamma} \right| < 0.1$ ; if the number of iterations is greater than 50, go to 4.

- 6''. Fix  $\gamma_n = \gamma_0$  and do a search over the interval  $[-4B, 4B]$  for that  $\alpha$  for which  $0 = \frac{\partial l}{\partial \alpha}$ ; set  $\alpha_n$  equal to the first value of  $\alpha$  for which  $\left| \frac{\partial l}{\partial \alpha} \right| < 0.1$ ; if the number is greater than 50, go to 4.

The following are empirical observations. The procedure involves a great deal of computing. For small times  $t$ , the estimate of  $W$  can be very close to zero; a lower bound of 0.1 was placed on the value of  $W$  that is input into the Kalman procedure. Further, a lower bound of  $-2$  was also put on  $\hat{\mu}_t$ . The stopping criteria of  $\max \left[ \left| \frac{\partial l}{\partial \alpha} \right|, \left| \frac{\partial l}{\partial \gamma} \right| \right] < 0.1$  seems to be adequate; a smaller tolerance can result in the procedure becoming unstable and greatly increasing the computational effort.

## 5. ESTIMATION OF $\alpha$ AND $W$ FOR THE POISSON/NORMAL MODEL: LOGARITHMIC TRANSFORMATION

Results of Lambert (1989) suggest that when count data arise from a mixture of Poisson distributions, then a smaller power transformation of the counts than the square root is needed to stabilize the variance of the count data. In particular, if the variability of the mixture is largish, then a log transformation may stabilize the variance. In this section simple procedures for estimating  $\alpha$  and  $W$  based on a log transformation of the count data are described.

Suppose  $Y_t$  and  $\mu_t$  are as in (1.1) and (1.2). Given  $\mu_{t+1}$ ,  $\beta$ , the first two terms of a Taylor expansion yield

$$\ln Y_{t+1} \approx \ln h_{t+1} + x_{t+1}\beta + \mu_{t+1} + [h_t \exp\{x_{t+1}\beta + \mu_{t+1}\}]^{-1} [Y_t - h_t \exp\{x_{t+1}\beta + \mu_{t+1}\}] \quad (5.1)$$

Let

$$Z_{t+1} = \ln \left[ Y_{t+1} + \frac{1}{2} \right] - \ln h_{t+1} - x_{t+1}\beta. \quad (5.2)$$

Using the first term of the Taylor expansion (5.1)

$$Z_{t+1} \approx \mu_{t+1} = \alpha \mu_t + \omega_{t+1} \approx \alpha m_t + \omega_{t+1}. \quad (5.3)$$

Hence, given  $[Y_0, Y_1, \dots, Y_t]$ , the distribution of  $Z_{t+1}$  has approximate mean  $\alpha m_t$  and variance  $W$ . We will approximate the distribution of  $Z_{t+1}$  with a normal distribution having mean  $\alpha m_t$  and variance  $W$ . A ln-likelihood function under this approximation is (up to addition of constants)

$$l(\alpha, W) = \sum_{t=1}^{T-1} -\frac{1}{2} \ln W - \frac{1}{2} (z_{t+1} - \alpha m_t)^2 \frac{1}{W}; \quad (5.4)$$

$$\frac{\partial l}{\partial \alpha} = \sum_{t=1}^{T-1} (z_{t+1} - \alpha m_t) m_t \frac{1}{W} = 0; \quad (5.5)$$

$$\frac{\partial l}{\partial W} = \sum_{t=1}^{T-1} -\frac{1}{2} \frac{1}{W} + \frac{1}{2} (z_{t+1} - \alpha m_t)^2 \frac{1}{W^2} = 0. \quad (5.6)$$

Thus,

$$\hat{\alpha}_L(T) = \frac{\sum_{t=1}^{T-1} (z_{t+1} m_t)}{\sum_{t=1}^{T-1} m_t^2} \quad (5.7)$$

$$\hat{W}_L(T) = \frac{1}{T-1} \sum_{t=1}^{T-1} (z_{t+1} - \hat{\alpha}_L(T) m_t)^2 \quad (5.8)$$

are the resulting estimates of  $\alpha$  and  $W$ .

Simulation experiments suggest that  $\hat{\alpha}_L$  and  $\hat{W}_L$  tend to have a large positive bias.

Another possibility is to estimate  $\alpha$  by

$$\hat{\alpha}_m(T) = \frac{1}{T} \sum_{t=1}^{T-1} m_{t+1} m_t / \sum_{t=1}^{T-1} m_t^2 \quad (5.9)$$

as in (3.7) of Section 3 and  $W$  by



$$\hat{W}_{m,L}(T) = \frac{1}{T-1} \sum_{t=1}^{T-1} (z_{t+1} - \hat{\alpha}_m(T)m_t)^2. \quad (5.10)$$

These estimates are easy to compute. Simulation experiments suggest that this estimate of  $\hat{\alpha}_m$  tends to have a negative bias and  $\hat{W}_{m,L}$  has a positive bias but not as large as that of  $\hat{W}_L$ .

## 6. RESULTS FROM SIMULATION EXPERIMENTS

In this section we report results of simulation experiments concerning the behavior of the approximate Kalman filter of Section 2 under various procedures for estimating  $\alpha$  and  $W$ .

Each replication of the simulation consists of generating a Poisson time series  $\{y_i; i = 0, 1, 2, \dots, T\}$  using equations (1.1) – (1.2). The random numbers were generated using LLRANDOM II, cf. Lewis et al. (1981). If  $\alpha < 1$ , then  $\mu_0$  is drawn from the stationary distribution of  $\{\mu_i; i \geq 0\}$ . The Kalman procedure with each of the procedures for estimating  $\alpha$  and  $W$  is then applied to  $\{y_i; i = 0, 1, 2, \dots, T\}$ . For each time  $t$ , the estimate

$$\hat{\lambda}_t = \exp\{h_t \exp\{\hat{\mu}_t + x_t \hat{\beta}\}\} \quad (6.1)$$

is computed and the mean square prediction error

$$\hat{e} = \frac{1}{T-1} \sum_{t=1}^{T-1} (y_{t+1} - \hat{\lambda}_t)^2 \quad (6.2)$$

is calculated. In addition the average estimated values of  $\alpha$  and  $W$  and  $\beta$  are computed:

$$m(\alpha) = \frac{1}{T} \sum_{t=1}^T \hat{\alpha}_t \quad (6.3)$$

$$m(W) = \frac{1}{T} \sum_{t=1}^T \hat{W}_t \quad (6.4)$$

$$m(\beta) = \frac{1}{T} \sum_{t=1}^T b_t \quad (6.5)$$

where  $\hat{\alpha}_t$  and  $\hat{W}_t$  are the estimates of  $\alpha$  and  $W$  obtained after the observation at time  $t$  and  $b_t$  is as in Section 1.

The simulation is replicated  $N$  times. If  $\hat{e}_i, m_i(\alpha), m_i(W), m_i(\beta)$  represent the summary statistics from the  $i^{\text{th}}$  replication, then the mean of the summary statistics are computed for the  $N$  replications; i.e.

$$\bar{e} = \frac{1}{N} \sum_{i=1}^N \hat{e}_i; \quad (6.5)$$

$$\sqrt{\text{var}[e]} = \frac{1}{N-1} \sum_{i=1}^N (\hat{e}_i - \bar{e})^2; \quad (6.6)$$

$$\bar{\alpha} = \frac{1}{N} \sum_{i=1}^N m_i(\alpha); \quad (6.7)$$

$$\bar{W} = \frac{1}{N} \sum_{i=1}^N m_i(W); \quad (6.8)$$

$$\bar{\beta} = \frac{1}{N} \sum_{i=1}^N m_i(\beta). \quad (6.8)$$

These latter statistics are then compared for the different procedures of estimating  $\alpha$  and  $W$ .

Results for the following procedures for estimating  $\alpha$  and  $W$  are reported.

1. DD:  $\alpha$  and  $W$  are both known and are not estimated.

2. FT: Both  $\alpha$  and  $W$  are estimated by a two-dimensional search of the likelihood based on the Freeman-Tukey transformation (4.9). The grid for  $W$  is  $[0.1,3]$  in increments of 0.1; the grid for  $\alpha$  is  $[-3,3]$  in increments of 0.1.
3. M/FT:  $\alpha$  is estimated by the moment estimator using (3.7);  $W$  is estimated by searching the one-dimensional likelihood based on the Freeman-Tukey transformation with  $\alpha$  set equal to (3.7); the grid is  $[0.1,3]$  with steps of size 0.1.
4. M/LN:  $\alpha$  is estimated using the moment estimate of (3.7).  $W$  is estimated using the moment estimator (5.10) based on the logarithmic transformation.
5. LN/LN:  $\alpha$  and  $W$  are estimated using the moment estimators (5.9) and (5.10) based on the logarithmic transformation.

All the procedures have a lower bound of  $-2$  on  $\hat{\mu}_r$ , a lower bound of  $0.1$  on  $\hat{W}_r$  and an upper bound of  $1$  on the absolute value of  $\hat{\alpha}_r$ .

For the results of the simulation experiments reported in Table 1,  $\alpha = 0.5$ ,  $W = 0.25$ ,  $x_t = 1$ ,  $h_t = 1$ ,  $\beta = 0.5$ . The initial values of the estimates  $\hat{W}_0 = 0.25$ ,  $\hat{\alpha}_0 = 0.5$ ,  $\hat{\mu}_0 = 0$ ; for the Kalman  $\hat{C}_0 = 1$ ,  $\hat{\tau}_0 = 1$ ,  $\hat{\rho}_0 = 0$ ,  $\hat{\beta} = 0$ . The two-dimensional search for the estimates of  $\alpha$  and  $W$  in FT takes a large amount of computer time. As a result, the number of replications for the experiments reported in Table 1 is small. However, the results of Table 1 suggest that the two most promising procedures are M/LN and M/FT.

In Table 2, the number of replications is 100. The procedures M/LN and M/FT only are compared. In Table 2, the initial values of  $\hat{W}_0 = 0.25$ ,  $\hat{\alpha}_0 = 0.5$ ,  $\hat{\mu}_0 = 0$ ; for the Kalman  $\hat{C}_0 = 1$ ,  $\hat{\tau}_0 = 1$ ,  $\hat{\rho}_0 = 0$ ,  $\hat{\beta} = 0$ .

In Table 3,  $\{x_t\} = \{0.25, 0.5, 1, 0.25, 0.5, 1, \dots\}$ ,  $\beta = 0.5$ ,  $W = 0.25$ ,  $\alpha = 0.5$ . The initial values of  $\hat{W}_0 = 0.25$ ,  $\hat{\alpha}_0 = 0.5$ ,  $\hat{\mu}_0 = 0$ ; for the Kalman  $\hat{C}_0 = 1$ ,  $\hat{\tau}_0 = 1$ ,  $\hat{\rho}_0 = 0$ ,  $\hat{\beta} = 0$ .

In Table 4, the parameters are the same as in Table 3 except  $W = 1$ . The initial values of  $\hat{W}_0 = 0.25$ ,  $\hat{\alpha}_0 = 0.5$ ,  $\hat{\mu}_0 = 0$ ; for the Kalman  $\hat{C}_0 = 1$ ,  $\hat{\tau}_0 = 1$ ,  $\hat{\rho}_0 = 0$ ,  $\hat{\beta} = 0$ .

Comparison of Tables 2 and 3 suggests, not surprisingly, that the mean square prediction error is smaller when there is a variable covariate  $\{x_i\}$ .

Comparison of Tables 3 and 4 suggests, not surprisingly, that a larger value of  $W$  results in a larger mean square prediction error.

TABLE 1.

SERIES LENGTH = 5; 20 REPLICATIONS

METHOD:	DD	FT	M/FT	M/LN	LN/LN
Mean MSE	4.34	4.91	4.77	4.19	4.96
St. Dev. MSE	3.21	3.82	3.64	3.17	3.76
Max MSE	11.0	13.2	11.7	12.5	13.5
Mean $\hat{\alpha}$	-	0.10	0.11	0.15	0.26
Mean $\hat{W}$	-	0.48	0.67	0.98	0.77

SERIES LENGTH = 10; 20 REPLICATIONS

METHOD:	DD	FT	M/FT	M/LN	LN/LN
Mean MSE	2.94	3.15	3.08	2.77	3.36
St. Dev. MSE	2.27	2.42	2.28	1.93	2.54
Max MSE	10.4	11.1	10.1	8.27	11.2
Mean $\hat{\alpha}$	-	0.12	0.08	0.09	0.19
Mean $\hat{W}$	-	0.23	0.32	0.78	0.69

SERIES LENGTH = 20; 40 REPLICATIONS

METHOD:	DD	FT	M/FT	M/LN	LN/LN
Mean MSE	3.38	3.90	3.72	3.29	3.97
St. Dev. MSE	2.00	2.81	2.43	2.16	2.62
Max MSE	10.7	13.5	12.1	10.3	11.2
Mean $\hat{\alpha}$	-	0.33	0.15	0.17	0.31
Mean $\hat{W}$	-	0.22	0.31	0.79	0.73

TABLE 2.  $x_t \equiv 1$ ,  $\alpha = 0.5$ ,  $W = 0.25$

SERIES LENGTH = 5; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	4.21	4.75	4.32
St. Dev. MSE	4.68	5.81	5.40
Max MSE	28.7	38.0	34.1
Mean $\hat{\alpha}$	-	0.08	0.09
Mean $\hat{W}$	-	0.55	0.93
Mean $\hat{\beta}$	0.34	0.25	0.23

SERIES LENGTH = 10; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	3.98	4.41	4.14
St. Dev. MSE	3.27	3.87	3.58
Max MSE	16.4	19.0	17.3
Mean $\hat{\alpha}$	-	0.07	0.10
Mean $\hat{W}$	-	0.45	0.88
Mean $\hat{\beta}$	0.42	0.36	0.31

SERIES LENGTH = 20; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	3.96	4.35	4.18
St. Dev. MSE	2.59	3.30	3.06
Max MSE	20.3	26.5	23.6
Mean $\hat{\alpha}$	-	0.10	0.13
Mean $\hat{W}$	-	0.41	0.82
Mean $\hat{\beta}$	0.48	0.43	0.37

TABLE 3. VARIABLE  $\{x_t\}$ ,  $\alpha = 0.5$ ,  $W = 0.25$

SERIES LENGTH = 5; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	3.69	3.89	3.70
St. Dev. MSE	3.83	4.09	4.00
Max MSE	15.6	18.4	19.9
Mean $\hat{\alpha}$	-	0.13	0.12
Mean $\hat{W}$	-	0.52	0.80
Mean $\hat{\beta}$	0.32	0.21	0.18

SERIES LENGTH = 20; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	3.11	3.32	3.21
St. Dev. MSE	2.48	2.80	2.68
Max MSE	20.4	22.7	20.2
Mean $\hat{\alpha}$	-	0.12	0.10
Mean $\hat{W}$	-	0.35	0.78
Mean $\hat{\beta}$	0.41	0.36	0.28

TABLE 4. VARIABLE  $\{x_t\}$ ,  $\alpha = 0.5$ ,  $W = 1$

SERIES LENGTH = 5; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	19.21	20.70	19.88
St. Dev. MSE	52.2	53.0	52.1
Max MSE	415.1	390.1	393.5
Mean $\hat{\alpha}$	-	0.19	0.18
Mean $\hat{W}$	-	0.93	1.05
Mean $\hat{\beta}$	0.31	0.14	0.12

SERIES LENGTH = 20; 100 REPLICATIONS

METHOD:	DD	M/FT	M/LN
Mean MSE	18.93	21.6	20.4
St. Dev. MSE	28.6	31.5	30.9
Max MSE	181.7	192.1	188.5
Mean $\hat{\alpha}$	-	0.20	0.19
Mean $\hat{W}$	-	1.2	1.1
Mean $\hat{\beta}$	0.55	0.32	0.31



Of the two procedures, M/LN tends to have the smaller mean MSE. M/LN tends to have a positive bias estimating  $W$  and negative biases estimating  $\hat{\alpha}$  and  $\hat{\beta}$ . The procedure M/FT takes more computational effort, tends to underestimate  $W$ ,  $\alpha$  and  $\beta$  and produces slightly higher mean MSE.

Tables 5-7 report results comparing procedures estimating  $\alpha$  and  $W$  with the procedure NFT in which both  $\alpha$  and  $W$  are estimated using the Newton procedure of Section 4. For all the procedures reported in these tables, there is no bounding on  $\hat{\alpha}$ . In Table 5,  $x_t \equiv 1$ ,  $h_t \equiv 1$ ,  $\beta = 0.5$ ,  $\alpha = 0.5$ ,  $W = 0.25$ . In Table 6,  $x_t = 0.25, 0.5, 1, 0.25, 0.5, 1, \dots$ , and the other parameters are as before. For Tables 5 and 6 the initial values of  $\hat{W}_0 = 0.25$ ,  $\hat{\alpha}_0 = 0.5$ ,  $\hat{\mu}_0 = 0$ ,  $\hat{C}_0 = 1$ ,  $\hat{\tau}_0 = 1$ ,  $\hat{\rho}_0 = 0$ . The values in parentheses are estimates of standard deviations; e.g.

$$\left[ \frac{1}{N-1} \sum_{i=1}^N (m_i(\alpha) - \bar{\alpha})^2 \right]^{\frac{1}{2}}.$$

For Table 7,  $\{x_t\}$  is as in Table 6,  $\alpha = 0.5$ ,  $W = 0.25$ ,  $\beta_0 = 0.5$ , and the initial estimates are  $\hat{\alpha}_0 = 0$ ,  $\hat{W}_0 = 1$ ,  $\hat{C}_0 = 1$ ,  $\hat{\tau}_0 = 1$ ,  $\hat{\rho}_0 = 0$ ,  $\hat{\mu}_0 = 0$ .

The Newton procedure of Section 4 does not take as much time as the two-dimensional search used in Tables 1-2; however, it is still a much larger computational effort than the other two procedures. Once again, based on the mean MSE and computational effort the procedure M/LN appears to be the most attractive. The procedure M/LN once again appears to overestimate  $W$  and underestimate  $\alpha$  and  $\beta$ .

TABLE 5.  $x_t \equiv 1$ ;  $\alpha = 0.5$ ,  $W = 0.25$ ,  $\beta = 0.5$

SERIES LENGTH = 5, 25 REPLICATIONS

METHOD:	DD	NFT	M/FT	M/LN
Mean MSE	3.94	4.44	4.32	3.89
Std Dev. MSE	3.08	3.66	3.49	3.10
Max MSE	11.0	13.2	11.7	12.5
Mean $\hat{\alpha}$	-	0.06 (0.37)	0.08 (0.22)	0.10 (0.23)
Mean $\hat{W}$	-	0.46 (0.47)	0.60 (0.60)	0.93 (0.45)
Mean $\hat{\beta}$	0.28	0.18	0.21	0.18

SERIES LENGTH = 20, 25 REPLICATIONS

METHOD:	DD	NFT	M/FT	M/LN
Mean MSE	3.11	3.40	3.32	3.14
Std Dev. MSE	1.83	2.12	2.11	2.03
Max MSE	8.50	9.29	9.60	8.78
Mean $\hat{\alpha}$	-	0.22 (0.39)	0.10 (0.30)	0.12 (0.29)
Mean $\hat{W}$	-	0.29 (0.24)	0.32 (0.26)	0.80 (0.23)
Mean $\hat{\beta}$	0.38	0.23	0.31	0.23

TABLE 6. VARIABLE  $\{x_t\}$ ;  $\alpha = 0.5$ ,  $W = 0.25$ ,  $\beta = 0.5$

SERIES LENGTH = 20, 25 REPLICATIONS

METHOD:	DD	NFT	M/FT	M/LN
Mean MSE	2.63	2.77	2.81	2.63
Std Dev. MSE	1.78	1.95	2.06	1.79
Max MSE	9.24	10.10	10.86	9.12
Mean $\hat{\alpha}$	-	0.21 (0.38)	0.06 (0.32)	0.06 (0.34)
Mean $\hat{W}$	-	0.20 (0.14)	0.25 (0.20)	0.68 (0.18)
Mean $\hat{\beta}$	0.33	0.21	0.27	0.18

TABLE 7. VARIABLE  $\{x_t\}$ ;  $\alpha = 0.5, W = 0.25, \beta = 0.5$

$$\hat{\alpha}_0 = 0, \hat{W}_0 = 1, \hat{C}_0 = 1, \hat{\tau}_0 = 1, \hat{\rho}_0 = 0, \hat{\mu}_0 = 0$$

SERIES LENGTH = 20, 100 REPLICATIONS

METHOD:	DD	NFT	M/FT	M/LN
Mean MSE	2.95	3.25	3.21	3.04
Std Dev. MSE	2.21	2.70	2.67	2.28
Max MSE	13.2	16.5	16.5	13.8
Mean $\hat{\alpha}$	-	0.12 (0.34)	0.11 (0.28)	0.09 (0.29)
Mean $\hat{W}$	-	0.34 (0.44)	0.36 (0.32)	0.73 (0.19)
Mean $\hat{\beta}$	0.36	0.25	0.28	0.21

## 7. SUMMARY AND CONCLUSIONS

In this paper we have investigated approximate methods for estimation and prediction for the hierarchical Poisson/normal time series model given in (1.1)–(1.2). For given values of the random walk parameters,  $\alpha$  and  $W$ , the joint distribution of  $(\boldsymbol{\beta}, \boldsymbol{\mu}_t)$  is approximated by a bivariate normal distribution using the Laplace method. Various heuristic methods for estimating  $\alpha$  and  $W$  are presented. Based on simulation results and ease of computation, it is recommended that  $\alpha$  be estimated by (3.7)

$$\hat{\alpha}(T) = \frac{\sum_{t=1}^T m_t m_{t-1}}{\sum_{t=1}^T m_{t-1}^2}$$

and  $W$  be estimated by (5.10)

$$\hat{W}(T) = \frac{1}{T-1} \sum_{t=1}^{T-1} (z_{t+1} - \hat{\alpha}(T)m_t)^2$$

where  $z_{t+1}$  is given by (5.2).

These estimates of  $\alpha$  and  $W$  along with the approximate Kalman procedure using the Laplace method provide a computationally easy procedure for prediction of the time series.

Figures 4-7 show results of a simulation of model (1.1)–(1.2) for  $t = 1, \dots, 100$ . In the example  $\beta = 0.5$ ,  $\{x_t\}$  takes the values  $\{0.25, 0.5, 1\}$  over and over, and  $h_t \equiv 1$ .  $\{\omega_t\}$  are iid normal with mean 0, variance 0.25, and  $\alpha = 0.5$ . The simulation starts with  $\boldsymbol{\mu}_0$  drawn from the stationary distribution of  $\{\boldsymbol{\mu}_t\}$ , a normal distribution with mean 0 and variance  $W/(1-\alpha^2) = \frac{1}{3}$ . The values of  $\alpha$  and  $W$  are estimated using (3.7) and (5.10). Initial values of  $\alpha$  and  $W$  are

$\alpha_0 = 0.5$  and  $W = 1, C_0 = 1, \tau_0 = 1, \rho_0 = 0, \beta_0 = 0, \mu_0 = 0$ . There are no bounds on  $\hat{\mu}_t, \hat{\alpha}$  and  $\hat{W}$ .

Figure 4 displays the count data. Also displayed are the true  $\lambda_t$  (circles), the estimated  $\lambda_t$  when both  $\alpha$  and  $W$  are known (dashed line) and the estimated  $\lambda_t$  when both  $\alpha$  and  $W$  are estimated (solid line). The  $\hat{\lambda}_t$  when both  $\alpha$  and  $W$  are estimated is perhaps a little more responsive to changes in the data than when  $\alpha$  and  $W$  are known. The difference between the two estimates of  $\lambda_t$  is greatest for small times  $t$ .

Figure 5 presents plots of the estimated  $\sqrt{W}$  and  $\alpha$ . Note that  $W$  has a positive bias which will make the Kalman procedure more responsive to the count data. The estimated values of  $\alpha$  appear to be reasonable.

Figure 6 presents estimates of  $\beta$  and its estimated standard deviation  $\sqrt{\tau_t}$  both for the Kalman with  $\alpha$  and  $W$  known (dotted line) and with  $\alpha$  and  $W$  unknown (solid line). The estimates of  $\tau_t$  generally decrease as  $t$  increases reflecting the model assumption that  $\beta$  is constant. The estimates of  $\beta$  appear to be reasonable. The estimates of  $\beta$  with  $\alpha$  and  $W$  also estimated tend to be smaller than those for which  $\alpha$  and  $W$  are known; this behavior may be due to the fact that estimation of  $\alpha$  and  $W$  is accounting for some of the variability of the data that would otherwise be accounted for by  $\beta$ .

Figure 7 displays the true  $\mu_t$  (circles), the estimated  $\hat{\mu}_t = m_t$  with parameters  $\alpha$  and  $W$  known (dotted line) and the estimated  $\hat{\mu}_t$  with parameters  $\alpha$  and  $W$  estimated (solid line). The count data are also displayed. The estimates of  $\hat{\mu}_t$  with  $\alpha$  and  $W$  estimated are more variable than those when  $\alpha$  and  $W$  are known reflecting the greater responsiveness of the Kalman to the

data when the estimated  $W$  has a position bias. Once again the estimates of  $\mu_i$  appear to be reasonable.

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COUNT DATA, TRUE LAMBDA, ESTIMATED LAMBDA AT T

COUNT DATA; DOT=TRUE LAM; SOLID=EST LAM

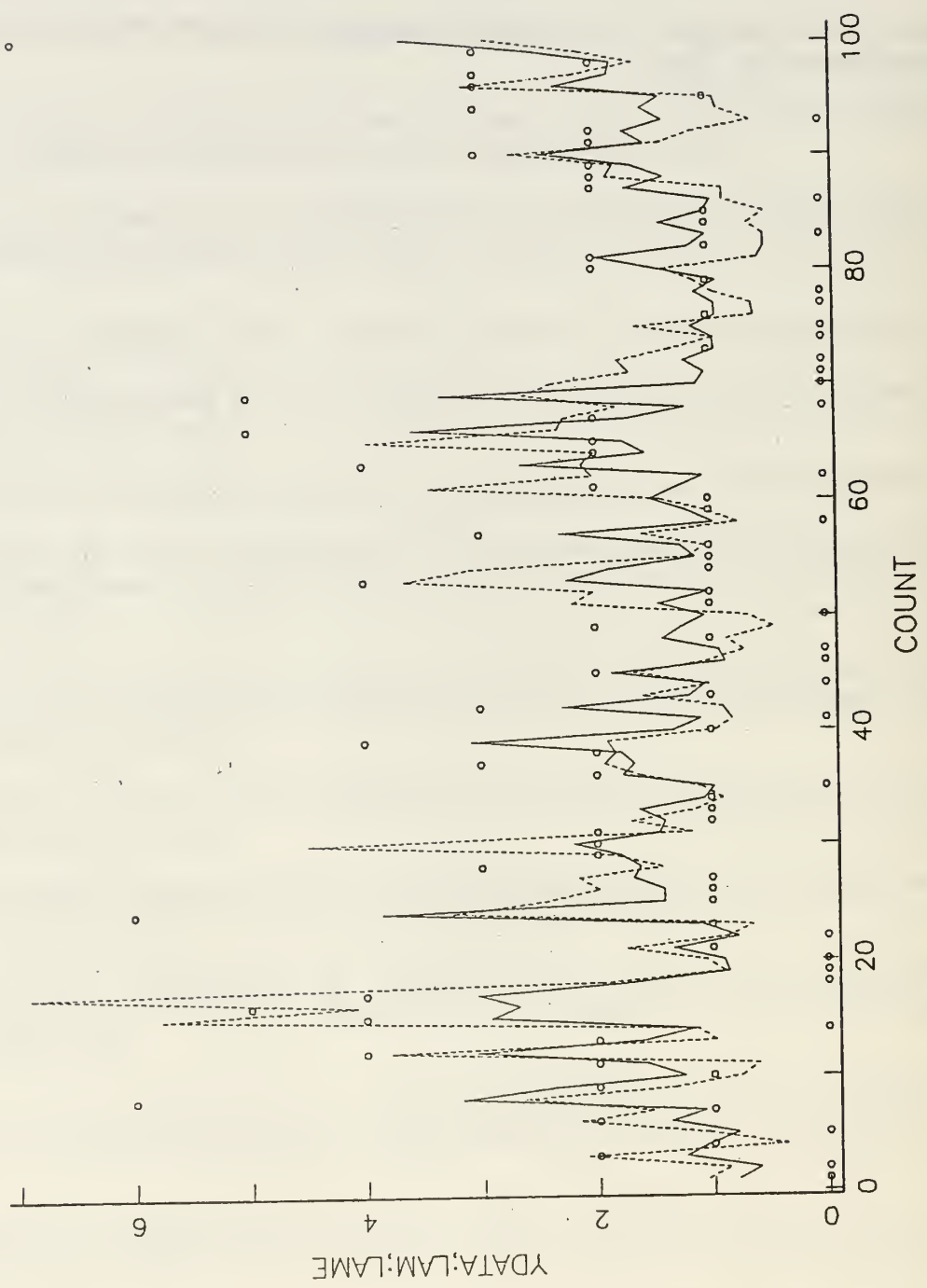
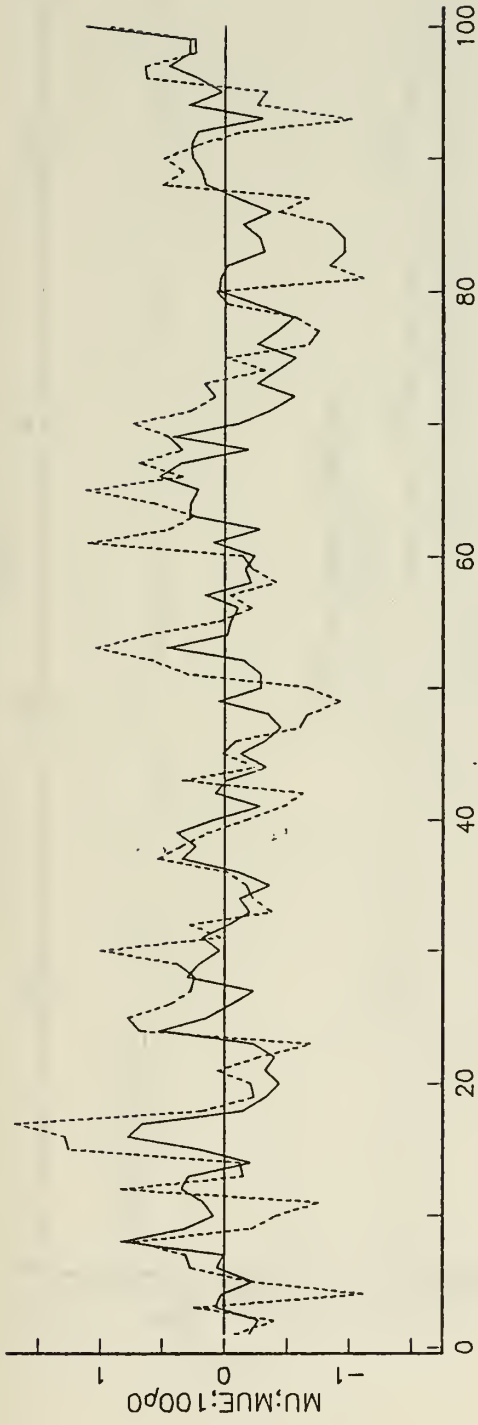


Figure 1

TRUE MU, ESTIMATED MU AT T

DOT=TRUE MU; SOLID=EST MU



SQUARE ROOT OF C[T]

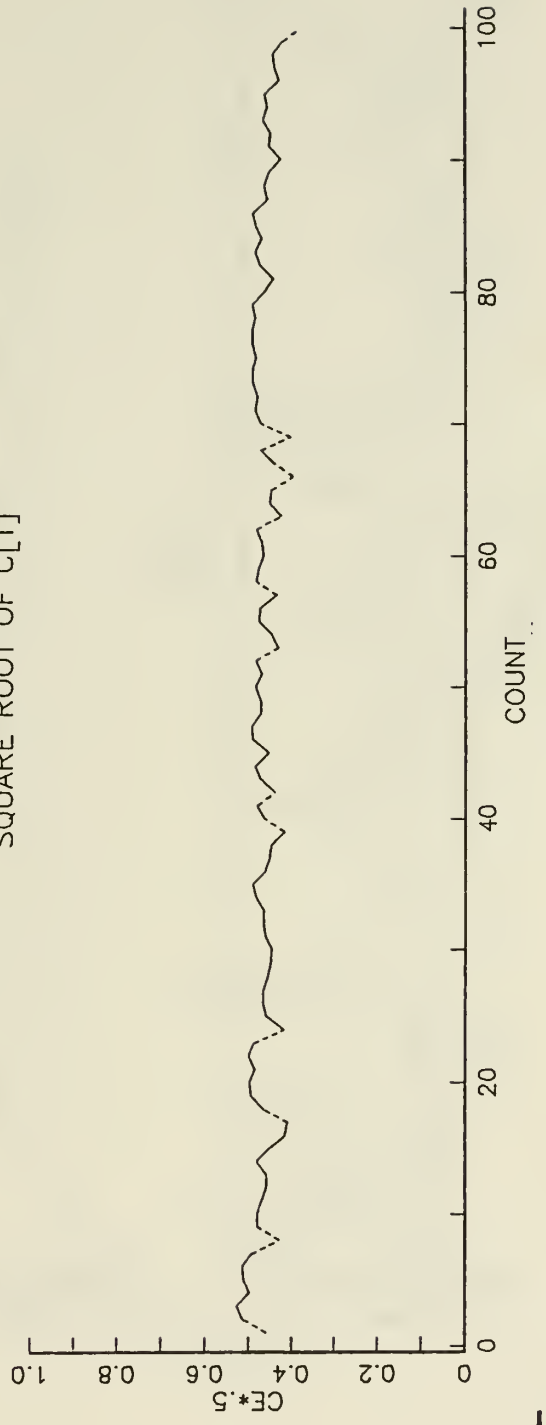


Figure 2

# ESTIMATED BETA

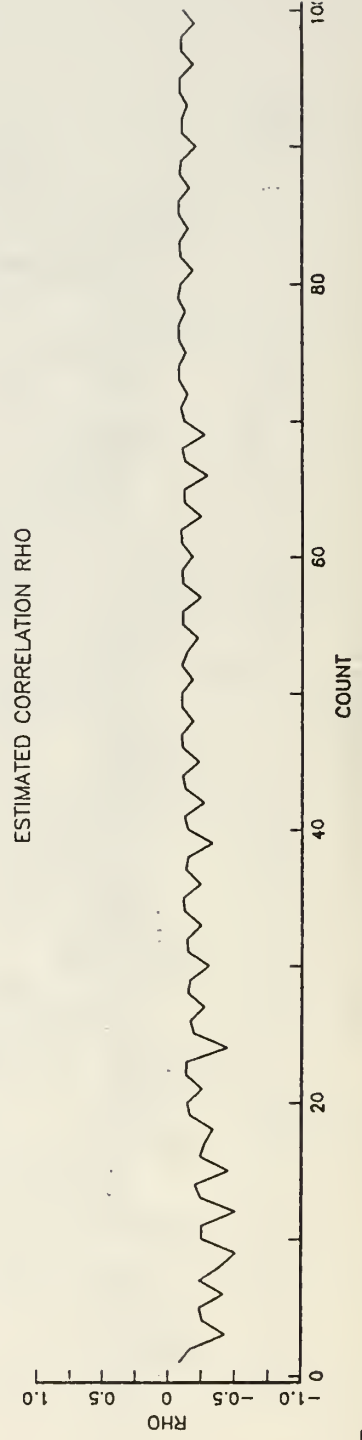
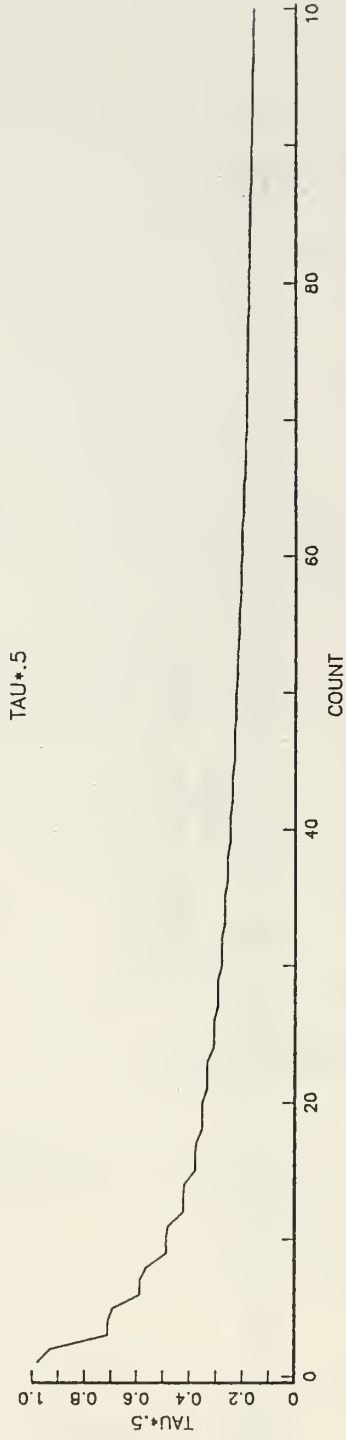
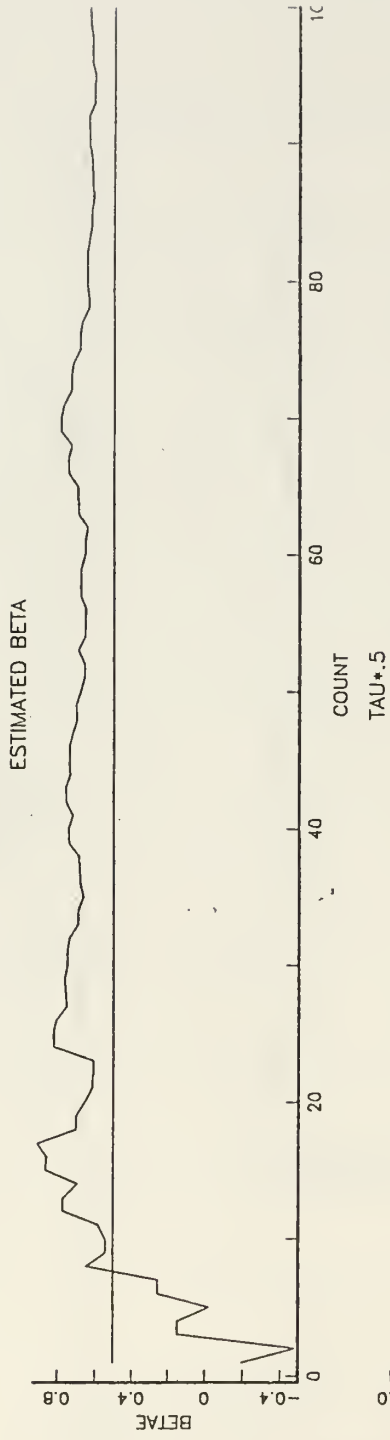
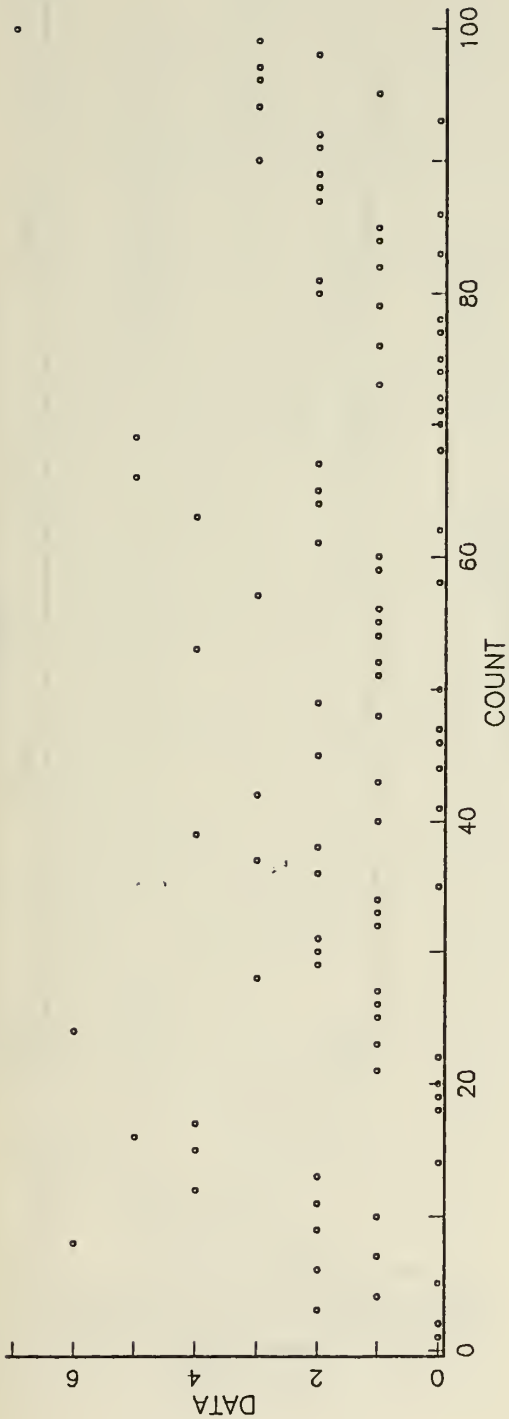


Figure 3

# ESTIMATED LAMBDA

COUNT DATA



TRUE LAM =  $\circ$ ; EST LAM: KNOWN PAR=DOTTED LINE; EST PAR=SOLID

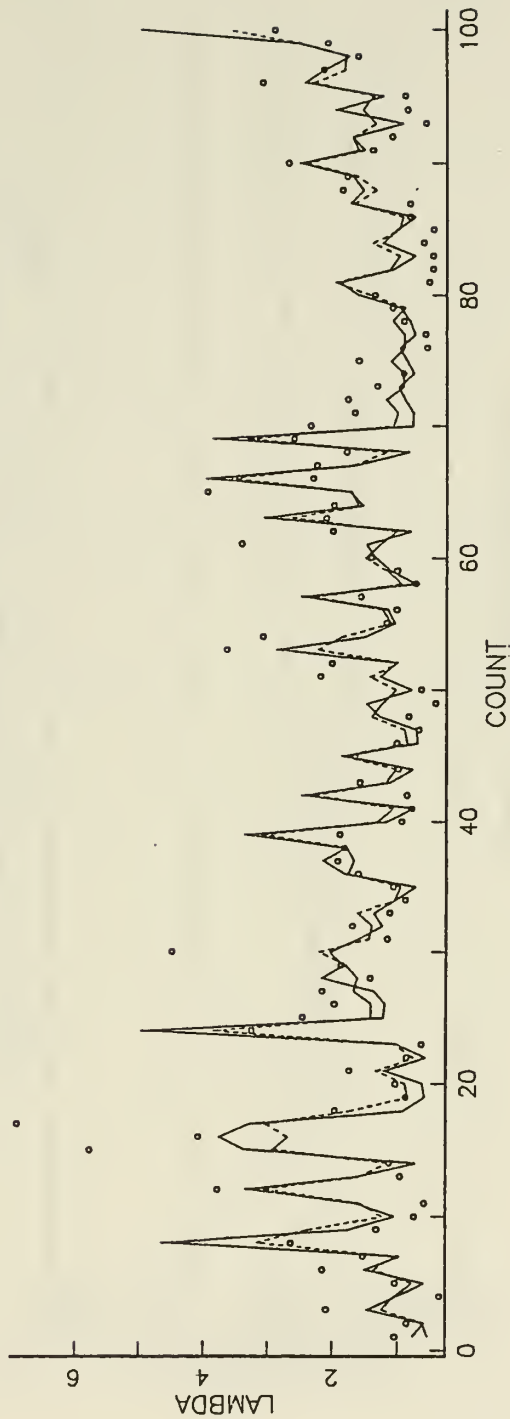
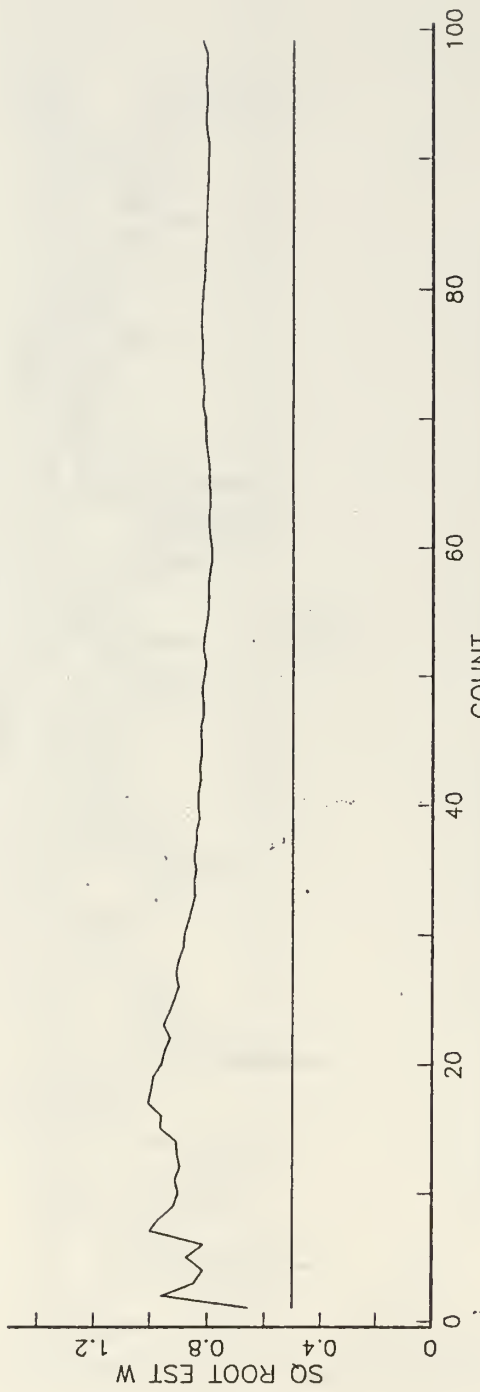


Figure 4

# COMPARISON OF EST AND TRUE PARAMETER ESTIMATES

SQUARE ROOT OF ESTIMATED W



ESTIMATED ALPHA

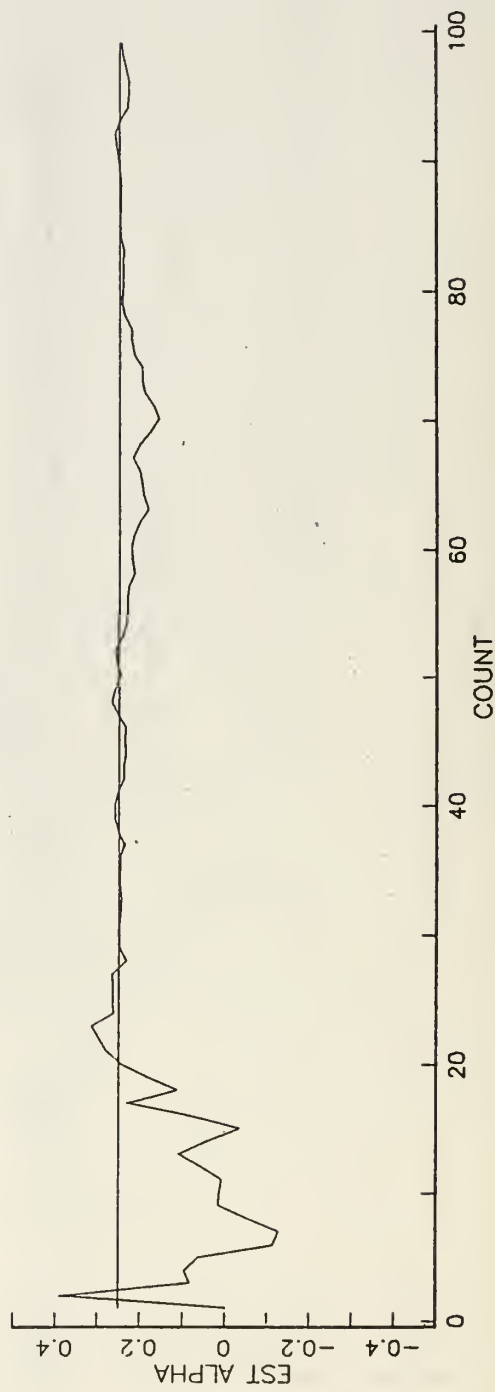
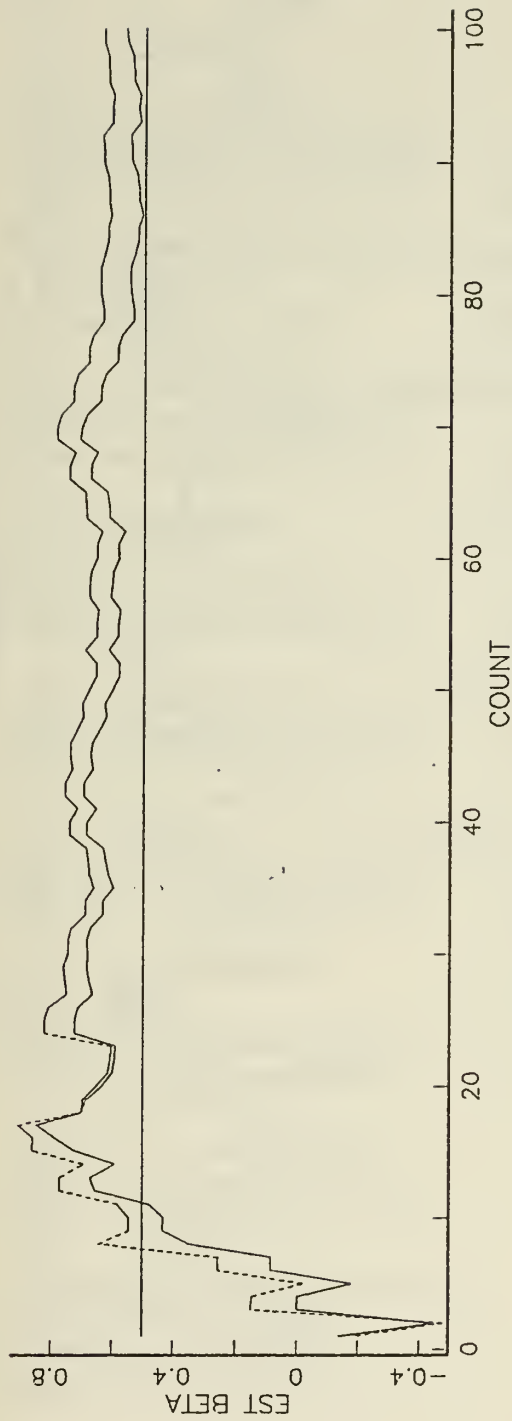


Figure 5

# ESTIMATES OF BETA

ESTIMATED BETA; KNOWN PAR=DOTTED; EST PAR=SOLID



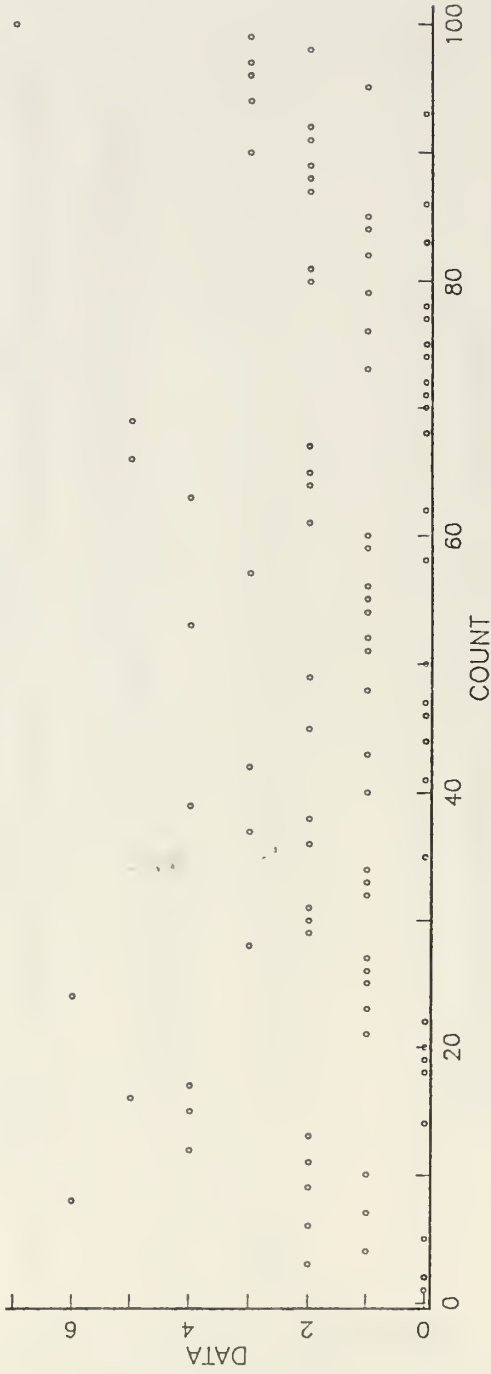
ESTIMATED TAU\*.5; KNOWN PAR=DOTTED; EST PAR=SOLID



Figure 6

# COMPARISON OF EST AND TRUE MU

COUNT DATA



TRUE MU=CIRCLES;EST MU KNOWN PAR=DOT LINE;EST MU=SOLID

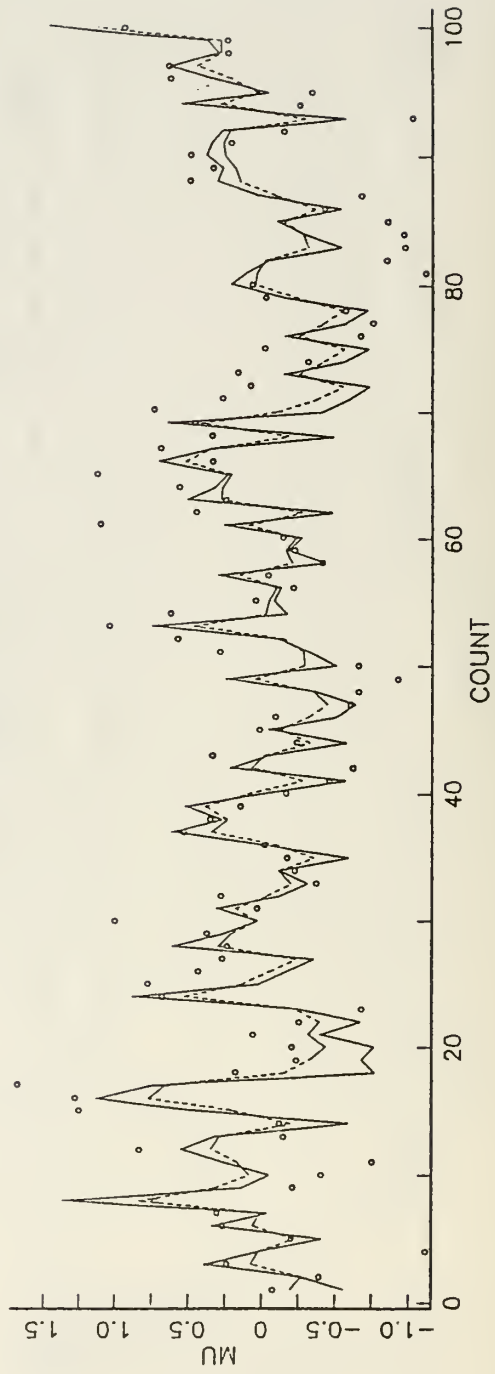


Figure 7



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